MSE, 이공계생을 위한 확률과 통계

[연습문제 답안 이용 안내]

Chapter 05 연습문제 해답

5.1

(a)

가능한 표본	$\bar{x}$
{1, 2, 3}	2
{1, 2, 4}	2.33
{2, 3, 4}	3

(b)

$\bar{x}$	$p(\bar{x})$
2	1/3
2.33	1/3
3	1/3

(c)

$$E(\bar{x}) = \sum \bar{x}p(\bar{x})$$

$$= 2\times\frac{1}{3} + 2.33\times\frac{1}{3} + 3\times\frac{1}{3} \fallingdotseq 2.443$$

(d)

$$Var(\bar{x}) = \sum \bar{x}^2 p(\bar{x}) - \bar{x}^2$$

$$= 2^2\times\frac{1}{3} + 2.33^2\times\frac{1}{3} + 3^2\times\frac{1}{3} - (2.443)^2$$

$$\fallingdotseq 0.1747$$

(e) $\dfrac{N-n}{N-1} = \dfrac{4-3}{4-1} = \dfrac{1}{3}$

5.3

표본평균 $\bar{x}$ 의 표준오차 $\sigma_{\bar{x}} = \dfrac{\sigma}{\sqrt{n}}$ 는 표본크기 n 이 커지면 작아지게 된다.

5.5

랜덤하게 추출한 10개월간의 평균을 $\overline{X}$라 하자. 그러면 $\overline{X}$는 $N\left(5650, \frac{700^2}{10}\right)$을 따른다고 할 수 있다. 따라서 $P\left(\left|\frac{\overline{X}-5650}{700/\sqrt{10}}\right| \le Z_{\alpha/2}\right) = 1-\alpha$이므로

$$\begin{aligned} P(|\overline{X}-\mu| \le 210) &= P\left(\left|\frac{\overline{X}-\mu}{700/\sqrt{10}}\right| \le 210\cdot\frac{\sqrt{10}}{700}\right) \\ &≒ P(|Z| \le 0.95) \\ &= 1-2P(Z > 0.95) ≒ 0.6578 \end{aligned}$$

5.7

신장의 표본평균을 $\overline{x}$라고 하면, 구하고자 하는 확률은 다음과 같다.

$$\begin{aligned} P(168 \le \overline{x} \le 172) &= P\left(\frac{168-170}{6/\sqrt{36}} \le Z \le \frac{172-170}{6/\sqrt{36}}\right) \\ &= P(-2 \le Z \le 2) = 0.9544 \end{aligned}$$

5.9

(a) $\overline{x}_1 + \overline{x}_2 \sim N\left(\mu_1+\mu_2, \quad \frac{\sigma_1^{\ 2}}{n_1}+\frac{\sigma_2^{\ 2}}{n_2}\right)$

(b) $\overline{x}_1 - \overline{x}_2 \sim N\left(\mu_1-\mu_2, \quad \frac{\sigma_1^{\ 2}}{n_1}+\frac{\sigma_2^{\ 2}}{n_2}\right)$

5.11

(a)

$$M_X(t) = \int_0^\infty e^{tx} \frac{1}{2^{\frac{n}{2}} \Gamma\left(\frac{n}{2}\right)} x^{\frac{n}{2}-1} e^{-\frac{x}{2}} dx$$

$$= \frac{1}{2^{\frac{n}{2}} \Gamma\left(\frac{n}{2}\right)} \int_0^\infty x^{\frac{n}{2}-1} e^{-\frac{x(1-2t)}{2}} dx$$

에서 $y = \dfrac{x(1-2t)}{2}$ 로 두면, $dx = \dfrac{2}{1-2t} dy$ $\left(t < \dfrac{1}{2}\right)$ 이므로

$$M_X(t) = \frac{1}{2^{\frac{n}{2}} \Gamma\left(\frac{n}{2}\right)} \int_0^\infty \left(\frac{2y}{1-2t}\right)^{\frac{n}{2}-1} e^{-y} \frac{2}{1-2t} dy$$

$$= \frac{1}{\Gamma\left(\frac{n}{2}\right) (1-2t)^{\frac{n}{2}}} \int_0^\infty y^{\frac{n}{2}-1} e^{-y} dy$$

$$= (1-2t)^{-\frac{n}{2}}$$

(b) $M_X{}'(t) = -\dfrac{n}{2}(1-2t)^{-\frac{n}{2}-1}(-2)$

$$= n(1-2t)^{-\frac{n}{2}-1}$$

$E(X) = M_X{}'(0) = n$

(c) $M_X{}''(t) = \left(-\dfrac{n}{2}\right)\left(-\dfrac{n}{2}-1\right)(1-2t)^{-\frac{n}{2}-2}(-2)^2$

$E(X^2) = M_X{}''(0)$

$$= 4\left(\frac{n}{2}\right)\left(\frac{n+2}{2}\right) = n(n+2)$$

$Var(X) = M_X{}''(0) - [M_X{}'(0)]^2$

$$= n(n+2) - n^2 = 2n$$

5.13

(a) $t-$ 분포의 확률밀도함수는

$$\begin{aligned} f(x) &= \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\,\Gamma\left(\frac{n}{2}\right)}\left(1+\frac{x^2}{n}\right)^{-\frac{(n+1)}{2}} \\ &= \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n}\,\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{1}{2}\right)}\left(1+\frac{x^2}{n}\right)^{-\frac{(n+1)}{2}} \\ &= \frac{1}{\sqrt{n}\,\mathrm{B}\left(\frac{n}{2},\frac{1}{2}\right)}\left(1+\frac{x^2}{n}\right)^{-\frac{(n+1)}{2}} \end{aligned}$$

이므로 $E(X)$는 다음과 같이 구해진다.

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x\frac{1}{\sqrt{n}\,\mathrm{B}\left(\frac{n}{2},\frac{1}{2}\right)}\left(1+\frac{x^2}{n}\right)^{-\frac{(n+1)}{2}} dx \\ &= \frac{1}{\sqrt{n}\,\mathrm{B}\left(\frac{n}{2},\frac{1}{2}\right)}\int_{-\infty}^{\infty} x\left(1+\frac{x^2}{n}\right)^{-\frac{(n+1)}{2}} dx \\ &= 0 \end{aligned}$$

(b)

$$\begin{aligned} E(X^2) &= \int_{-\infty}^{\infty} x^2\frac{1}{\sqrt{n}\,\mathrm{B}\left(\frac{n}{2},\frac{1}{2}\right)}\left(1+\frac{x^2}{n}\right)^{-\frac{(n+1)}{2}} dx \\ &= \frac{1}{\sqrt{n}\,\mathrm{B}\left(\frac{n}{2},\frac{1}{2}\right)}\int_{-\infty}^{\infty} x^2\left(1+\frac{x^2}{n}\right)^{-\frac{(n+1)}{2}} dx \\ &= \frac{2}{\sqrt{n}\,\mathrm{B}\left(\frac{n}{2},\frac{1}{2}\right)}\int_{0}^{\infty} x\left(1+\frac{x^2}{n}\right)^{-\frac{(n+1)}{2}} xdx \end{aligned}$$

이므로, 이때 $\left(1+\frac{x^2}{n}\right)^{-1}=t$로 두면 $x^2=n(1-t)t^{-1}$, $xdx=\left(-\frac{n}{2t^2}\right)dt$ 이다. 따라서

$$
\begin{aligned}
E(X^2) &= \frac{2}{\sqrt{n}\,\mathrm{B}\left(\frac{n}{2}, \frac{1}{2}\right)} \int_1^0 \sqrt{n}\,(1-t)^{\frac{1}{2}} t^{-\frac{1}{2}} \cdot t^{\frac{n+1}{2}} \left(-\frac{n}{2t^2}\right) dt \\
&= \frac{n^{\frac{3}{2}}}{\sqrt{n}\,\mathrm{B}\left(\frac{n}{2}, \frac{1}{2}\right)} \int_0^1 t^{\frac{n}{2}-1-1} (1-t)^{\frac{1}{2}+1-1} dt \\
&= \frac{n}{\mathrm{B}\left(\frac{n}{2}, \frac{1}{2}\right)} \mathrm{B}\left(\frac{n}{2}-1, \frac{3}{2}\right), \quad \left(\frac{n}{2}-1 \text{ 에서 } n>2\right) \\
&= n \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n}{2}\right)} \frac{\Gamma\left(\frac{n}{2}-1\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} \\
&= n \frac{\Gamma\left(\frac{n}{2}-1\right)\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\left(\frac{n}{2}-1\right)\Gamma\left(\frac{n}{2}-1\right)} \\
&= \frac{n}{n-2}
\end{aligned}
$$

이 된다. 그러므로, $Var(X)$는 다음과 같이 구해진다.

$$
\begin{aligned}
Var(X) &= E(X^2) - [E(X)]^2 \\
&= \frac{n}{n-2} - 0^2 \\
&= \frac{n}{n-2}
\end{aligned}
$$

5.15

자유도가 ($n_1 - 1$, $n_2 - 1$)인 F분포이다.

5.17

표본비율 $\hat{p} = \frac{X}{n} \approx N\left(0.9, \frac{0.9 \times 0.1}{100}\right)$을 따른다. 그러므로

$Z = \frac{\hat{p} - 0.9}{0.03} \sim N(0,1)$을 따르게 된다.

따라서

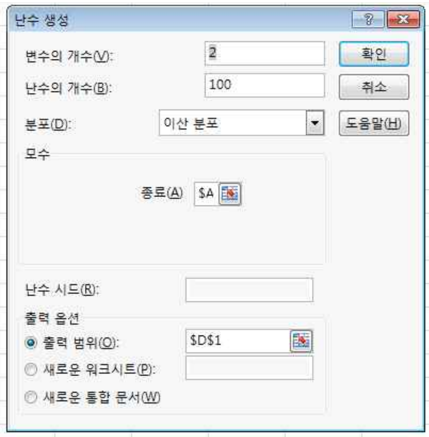
$$
\begin{aligned}
P(|\hat{p} - 0.9| \le 0.05) &= P\left(\frac{|\hat{p} - 0.9|}{0.03} \le \frac{5}{3}\right) = P\left(|Z| \le \frac{5}{3}\right) \\
&\fallingdotseq 2 \times 0.9525 - 1 = 0.905
\end{aligned}
$$

5.19

먼저 다음과 같이 자료를 입력한다.

	A	B
1	X	P(X)
2	1	0.25
3	2	0.25
4	3	0.25
5	4	0.25

[데이터분석]⇒[난수생성]을 통해 난수를 생성한다.



이때 분포는 이산분포를 선택하고 종료(A)에 위에서 입력한 내용을 선택하여 설정해준다. 그러면 원하는 결과를 얻을 수 있다.